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Abstract: The problem of finding a maximum cardinality matching in a d-partite, d-uniform hypergraph is an important problem in combinatorial optimization and has been theoretically analyzed. We first generalize some graph matching heuristics for this problem. We then propose a novel heuristic based on tensor scaling to extend the matching via judicious hyperedge selections. Experiments on random, synthetic and real-life hypergraphs show that this new heuristic is highly practical and superior to the others on finding a matching with large cardinality.

Key-words: d-dimensional matching, tensor scaling, matching in hypergraphs, Karp-Sipser heuristic

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Heuristiques efficaces pour le couplage dans des hypergraphes

Résumé : Le problème consistant à trouver un couplage maximal dans un hypergraphe uniforme ayant d parts est un problème important en optimisation combinatoire. Nous généralisons d'abord quelques heuristiques de couplage dans des graphes pour ce problème. Ensuite, nous proposons une nouvelle heuristique basée sur des méthodes de mise à l'échelle de tenseur pour étendre le couplage via des sélections judicieuses d'hyperarêtes. Des expériences sur des hypergraphes aléatoires, synthétiques et réels montrent que cette nouvelle heuristique est simple à mettre en pratique et supérieure aux autres pour trouver des couplages de grande cardinalité.

Mots-clés : couplage, hypergraphes, heuristique Karp–Sipser

1 Introduction

A hypergraph $H = (V, E)$ consists of a finite set V and a collection E of subsets of V. The set V is called vertices, and the collection E is called hyperedges. A hypergraph is called *d-partite* and *d-uniform*, if $V = \bigcup_{i=1}^{d} V_i$ with disjoint V_is and every hyperedge contains a single vertex from each V_i. A matching in a hypergraph is a set of disjoint hyperedges. In this paper, we investigate effective heuristics for finding large matchings in d-partite, d-uniform hypergraphs.

Finding a maximum cardinality matching in a d-partite, d-uniform hypergraph for $d \geq 3$ is NP-Complete; the 3-partite case is called the MAX-3-DM problem [26]. This problem has been studied mostly in the context of local search algorithms [24], and the best known algorithm is due to Cygan [9] who presents a $((d + 1 + \varepsilon)/3)$-approximation, building on previous work [10, 21]. It is also shown that it is NP-Hard to approximate MAX-3-DM within 98/97 [3]. Similar bounds exist for higher dimensions: the hardness of approximation for $d = 4, 5$ and 6 are shown to be $54/53 - \varepsilon$, $30/29 - \varepsilon$, and $23/22 - \varepsilon$, respectively [22].

Finding a maximum cardinality matching in a d-partite, d-uniform hypergraph is a special case of the d-SET-PACKING problem [23]. It has been shown that d-SET-PACKING is hard to approximate within a factor of $\mathcal{O}(d/\log d)$ [23]. The maximum/perfect set packing problem has many applications, including combinatorial auctions [20] and personnel scheduling [18]. Such a matching can also be used in the coarsening phase of multilevel hypergraph partitioning tools [6], when the input is d-uniform and d-partite. Such hypergraphs are used in modeling and partitioning tensors [28].

Our contributions in this paper are as follows. We propose five heuristics: The first two heuristics are adaptations of the well-known greedy [15] and Karp-Sipser [27] heuristics proposed for bipartite graph maximum cardinality matching. We use Greedy^g and Karp-Sipser^g to refer to these heuristics, and Greedy and Karp-Sipser for the proposed generalizations. Greedy traverses the hyperedge list in random order and adds an edge to the matching whenever possible. Karp-Sipser introduces certain rules to Greedy to improve the cardinality. The third heuristic is inspired by a recent scaling-based approach proposed for the maximum cardinality matching problem on graphs [12–14]. The fourth heuristic is a modification on the third one that allows for faster execution time. The last one finds a matching for a reduced, $(d - 1)$-dimensional problem and exploits it for d-dimensions. This heuristic uses an exact algorithm for the bipartite matching problem. We perform experiments to evaluate the performance of these heuristics on special classes of random hypergraphs as well as real-life data.

One plausible way to tackle the problem is to create the line graph G for a given hypergraph H. The line graph is created by identifying each hyperedge of H with a vertex in G, and by connecting two vertices of G with an edge, iff the corresponding hyperedges share a common vertex in H. Then, successful heuristics for computing large independent sets in graphs, e.g., KaMIS [29], can be used to compute large matchings in hypergraphs. This approach, although promising quality-wise, could be impractical. This is so, since building G from H requires quadratic run time (in terms of the number of hyperedges) and more importantly quadratic storage (again in terms of the number of hyperedges) in the worst case. While this can be acceptable in some instances, in some others it is not. We have such instances in the experiments. Notice that while a heuristic for the independent set problem can be of linear time complexity in graphs, due to our graphs being a line graph, the actual complexity could be high.

The rest of the paper is organized as follows. Section 2 introduces the notation and summarizes the background material. The proposed heuristics are summarized in Section 3. Section 4 presents the experimental results and Section 5 concludes the paper.

2 Background and notation

Tensors are multidimensional arrays, generalizing matrices to higher orders. Let $\mathbf{T}$ be a d-dimensional tensor whose size is $n_1 \times \cdots \times n_d$. The elements of $\mathbf{T}$ are shown with $\mathbf{T}_{i_1,\ldots,i_d}$, where $i_j \in \{1,\ldots,n_j\}$. A marginal is a $(d-1)$-dimensional section of a d-dimensional tensor, obtained by fixing one of its indices. A d-dimensional tensor where the entries in each of its marginals sum to one is called d-stochastic. In a d-stochastic tensor, all dimensions necessarily have the same size n. A d-stochastic tensor where each marginal contains exactly one nonzero entry (equal to one) is called a permutation tensor. Franklin and Lorenz [16] show that if a nonnegative tensor $\mathbf{T}$ has the same zero-pattern as a d-stochastic tensor $\mathbf{B}$, then one can find a set of d vectors $x^{(1)}, x^{(2)}, \ldots, x^{(d)}$ such that $\mathbf{T}_{i_1,\ldots,i_d} \cdot x^{(1)}_{i_1} \cdot \cdots \cdot x^{(d)}_{i_d} = \mathbf{B}_{i_1,\ldots,i_d}$ for all $i_1,\ldots,i_d \in \{1,\ldots,n\}$. In fact, a multidimensional version of the algorithm for doubly-stochastic scaling (of matrices) by Sinkhorn and Knopp [32] can be used to obtain these d vectors.

A d-partite, d-uniform hypergraph $H = (V_1 \cup \cdots \cup V_d, E)$ can be naturally represented by a d-dimensional tensor. This is done by associating each tensor dimension to a vertex class. Let $|V_i| = n_i$. Let the tensor $\mathbf{T} \in \{0,1\}^{n_1 \times \cdots \times n_d}$ have a nonzero element $\mathbf{T}_{v_1,\ldots,v_d}$ iff $(v_1,\ldots,v_d)$ is an edge of H. Then, $\mathbf{T}$ is called the adjacency tensor of H. We will use this correspondence for our third heuristic which enhances Karp-Sipser with tensor-scaling to improve the matching cardinality. In H, if a vertex is a member of only a single hyperedge we call it a degree-1 vertex. Similarly, if it is a member of only two we call it a degree-2 vertex.

In the *k-out random hypergraph model*, given V, each vertex $u \in V$ selects k hyperedges from the set $E_u = \{e : e \subseteq V, u \in e\}$ in a uniformly random fashion and the union of these edges forms E. We are interested in the d-partite, d-uniform case, and hence $E_u = \{e : |e \cap V_i| = 1 \text{ for } 1 \leq i \leq d, u \in e\}$. This model generalizes the random k-out bipartite graphs [34]. Devlin and Kahn [11] investigate fractional matchings in these hypergraphs, and mention in passing that k should be exponential in d to ensure that a perfect matching exists.

3 Heuristics for maximum d-dimensional matching

A matching which cannot be extended with more edges is called *maximal*. In this work, we propose heuristics for finding maximal matchings on d-partite, d-uniform hypergraphs. For such hypergraphs, any maximal matching is a d-approximate matching. The bound is tight and can be verified for $d = 3$. Let H be a 3-partite $3 \times 3 \times 3$ hypergraph with the following edges $e_1 = (1,1,1), e_2 = (2,2,2), e_3 = (3,3,3)$ and $e_4 = (1,2,3)$. The maximum matching is $\{e_1, e_2, e_3\}$ but the edge $\{e_4\}$ alone forms a maximal matching.

3.1 A Greedy heuristic for Max-d-DM

There exist two variants of Greedy^g proposed for graph matching in the literature. The first one [15] randomly visits the edges whereas the second one randomly visits the vertices [30]. We adapt the first variant to our problem and call it Greedy. It traverses the hyperedges in random order and adds the current hyperedge to the matching whenever possible. Since any maximal matching is possible as its output, Greedy is a d-approximation heuristic. It provides matchings of varying quality, depending upon the order in which the hyperedges are processed.

3.2 Karp-Sipser for Max-d-DM

A widely-used heuristic to obtain a (maximal) matching in graphs is Karp-Sipser^g [27]. On a graph, the heuristic iteratively adds a random edge to the matching and reduces the graph by removing its endpoints, as well as their edges. Whenever possible, Karp-Sipser^g does not apply a random selection but reduces the problem size, i.e., number of vertices in the graph by one via two rules:

- At any time during the heuristic, if a degree-1 vertex appears it is matched with its only neighbor.
- Otherwise, if a degree-2 vertex u appears with neighbors $\{v, w\}$, u (and its edges) is removed from the current graph, and v and w are merged to create a new vertex vw whose set of neighbors is the union of those of v and w (except u). A maximum cardinality matching for the reduced graph can be extended to obtain one for the current graph by matching u with either v or w depending on vw's match.

Both rules are optimal in the sense that they do not reduce the cardinality of a maximum matching in the current graph they are applied on. We now propose an adaptation of Karp-Sipser^g for d-partite, d-uniform hypergraphs. Similar to the original one, the modified heuristic iteratively adds a random hyperedge to the matching, remove its d endpoints, as well as their hyperedges. However, the random selection is not applied whenever hyperedges defined by the following lemmas appear.

Lemma 1. *During the heuristic, if a hyperedge e with at least $d-1$ degree-1 endpoints appears, there exists a maximum cardinality matching in the current hypergraph containing e.*

Proof. Let H' be the current hypergraph at hand and $e = (u_1, \ldots, u_d)$ be a hyperedge in H' whose first $d-1$ endpoints are degree-1 vertices. Let M' be a maximum cardinality matching in H'. If $e \in M'$, we are done. Otherwise, assume that u_d is the endpoint matched by a hyperedge $e' \in M'$ (note that if u_d is not matched M' can be extended with e). Since u_i, $1 \le i < d$, are not matched in M', $M' \setminus \{e'\} \cup \{e\}$ defines a valid maximum cardinality matching for H'. □

We note that it is not possible to relax the condition by using a hyperedge e with less than $d-1$ endpoints of degree-1; in M', two of e's higher degree endpoints could be matched with two different hyperedges, in which case the substitution as done in the proof of the lemma is not valid.

Lemma 2. *During the heuristic, let $e = (u_1, \ldots, u_d)$ and $e' = (u'_1, \ldots, u'_d)$ be two hyperedges sharing at least one endpoint where for an index set $\mathcal{I} \subset \{1, \ldots, d\}$ of cardinality $d-1$, the vertices u_i, u'_i for all $i \in \mathcal{I}$ only touch e and/or e'. That is for each $i \in \mathcal{I}$, either $u_i = u'_i$ is a degree-2 vertex or $u_i \ne u'_i$ and they are both degree-1 vertices. For $j \notin \mathcal{I}, u_j$ and u'_j are arbitrary vertices. Then, in the current hypergraph, there exists a maximum cardinality matching having either e or e'.*

Proof. Let H' be the current hypergraph at hand and $j \notin \mathcal{I}$ be the remaining part id. Let M' be a maximum cardinality matching in H'. If either $e \in M'$ or $e' \in M'$, we are done. Otherwise, u_i and u'_i for all $i \in \mathcal{I}$ are unmatched by M'. Furthermore, since M' is maximal, u_j must be matched by M' (otherwise, M' can be extended by e). Let $e'' \in M'$ be the hyperedge matching u_j. Then $M' \setminus \{e''\} \cup \{e\}$ defines a valid maximum cardinality matching for H'. □

Whenever such hyperedges appear, the rules below are applied in the same order:

- **Rule-1**: At any time during the heuristic, if a hyperedge e with at least $d-1$ degree-1 endpoints appears, instead of a random edge, e is added to the matching and removed from the hypergraph.
- **Rule-2**: Otherwise, if two hyperedges e and e' as defined in Lemma 2 appear, they are removed from the current hypergraph with the endpoints u_i, u'_i for all $i \in \mathcal{I}$. Then, we consider u_j and u'_j. If u_j and u'_j are distinct, they are merged to create a new vertex $u_ju'_j$, whose hyperedge list is defined as the union of u_j's and u'_j's hyperedge lists. If u_j and u'_j are identical, we rename u_j as $u_ju'_j$. After obtaining a maximal matching on the reduced hypergraph, depending on the hyperedge matching $u_ju'_j$, either e or e' can be used to obtain a larger matching in the current hypergraph.

When Rule-2 is applied, the two hyperedges identified in Lemma 2 are removed from the hypergraph, and only the hyperedges containing u_j and/or u'_j have an update in their vertex list. Since the original hypergraph is d-partite and d-uniform, that update is just a renaming of a vertex in the concerned hyperedges (hence the resulting hypergraph is d-partite and d-uniform).

Although the extended rules usually lead to improved results in comparison to Greedy, Karp-Sipser still adheres to the d-approximation bound of maximal matchings. To see this, we can use the toy example given as a worst-case for Greedy. For the example given at the beginning of Section 3, Karp-Sipser generates a maximum cardinality matching by applying the first rule. However, when $e_5 = (2, 1, 3)$ and $e_6 = (3, 1, 3)$ are added to the example, neither of the two rules can be applied. As before, in case e_4 is randomly selected, it alone forms a maximal matching.

3.3 Karp-Sipser-scaling for Max-d-DM

Karp-Sipser can be modified for better decisions in case neither of the two rules hold. In our variant, instead of a random selection, we first scale the adjacency tensor of H and obtain an approximate d-stochastic tensor $\mathbf{T}$. We then augment the matching by adding the edge which corresponds to the largest value in $\mathbf{T}$. The modified heuristic is summarized in Algorithm 1.

Our inspiration comes from the $d = 2$ case and more specifically from the relation between scaling and matching. It is known due to Birkhoff [4] that the polytope of $n \times n$ doubly stochastic matrices is the convex hull of the $n \times n$ permutation matrices. A nonnegative matrix $\mathbf{A}$ where all entries participate in some perfect matching can be scaled with two positive diagonal matrices $\mathbf{R}$ and $\mathbf{C}$ such that $\mathbf{RAC}$ is doubly stochastic. Otherwise, provided that $\mathbf{A}$ has a perfect matching, it can still be scaled to a doubly stochastic form, but not with these positive diagonal matrices. In this case, the entries not participating in any perfect matching tend to be zero in the scaled matrix. This fact is exploited to design randomized approximation algorithms for graph matching [12, 13]. By using the scaling as a preprocessing step and choosing edges with a probability corresponding to the scaled entry, the edges which are not included in a perfect matching become less likely to be chosen. The current algorithm differs from these approaches by selecting a single hyperedge at each step and applying scaling again before the next selection.

Unfortunately for $d \geq 3$, there is no equivalent of Birkhoff's theorem as demonstrated by the following lemma.

Lemma 3. *For $d \geq 3$, there exist extreme points in the set of d-stochastic tensors which are not permutation tensors.*

Algorithm 1 Karp-Sipser-scaling

Input: A d-partite d-uniform $n_1 \times \cdots \times n_d$ hypergraph $H = (V, E)$
Output: A maximal matching M of H

$M \leftarrow \emptyset$ ▶ Initially M is empty
$S \leftarrow \emptyset$ ▶ Stack for the merges for Rule-2
while H is not empty **do**
  Remove the isolated vertices from H
  if $\exists e = (u_1, \ldots, u_d)$ as in Rule-1 **then**
    $M \leftarrow M \cup \{e\}$ ▶ Add e to the matching
    Apply the reduction for Rule-1 on H
  else if $\exists e = (u_1, \ldots, u_d), e' = (u'_1, \ldots, u'_d)$ and $\mathcal{I}$ as in Rule-2 **then**
    Let j be the part index where $j \notin \mathcal{I}$
    Apply the reduction for Rule-2 on H by introducing the vertex $u_j u'_j$
    $E' = \{(v_1, \ldots, u_j u'_j, \ldots, v_d) : \text{for all } (v_1, \ldots, u_j, \ldots, v_d) \in E\}$
      ▶ memorize the hyperedges of u_j
    S.push(e, e', $u_j u'_j$, E') ▶ Store the current merge
  else
    $\mathbf{T} \leftarrow \textsc{Scale}(adj(H))$ ▶ Scale the adjacency tensor of H
    $e \leftarrow \arg\max_{(u_1, \ldots, u_d)} (\mathbf{T}_{u_1, \ldots, u_d})$ ▶ Find the maximum entry in $\mathbf{T}$
    $M \leftarrow M \cup \{e\}$ ▶ Add e to the matching
    Remove all hyperedges of $u_1, \ldots, u_d$ from E
    $V \leftarrow V \setminus \{u_1, \ldots, u_d\}$
while $S \neq \emptyset$ **do**
  $(e, e', u_j u'_j, E') \leftarrow S$.pop() ▶ Get the most recent merge
  if $u_j u'_j$ is not matched by M **then**
    $M \leftarrow M \cup \{e\}$
  else
    Let $e'' \in M$ be the hyperedge matching $u_j u'_j$
    if $e'' \in E'$ **then**
      Replace $u_j u'_j$ in e'' with u'_j
      $M \leftarrow M \cup \{e'\}$
    else
      Replace $u_j u'_j$ in e'' with u_j
      $M \leftarrow M \cup \{e\}$

Proof. We provide a $2 \times 2 \times 2$ tensor $\mathbf{T}^3$ with an inspiration from [8]. For convenience, we depict $\mathbf{T}^3$ by two 2×2 matrices as follows which are the marginals of the 3rd dimension:

$$\mathbf{T}^3_{:,:,1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \text{ and } \mathbf{T}^3_{:,:,2} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

The maximum matching cardinality in this tensor is 1 and it cannot be written as a linear combination of permutation tensors. This particular extreme point can be extended for higher d by setting $\mathbf{T}^d_{u_1,u_2,u_3,\ldots,u_3} = \mathbf{T}^3_{u_1,u_2,u_3}$ for each nonzero element $\mathbf{T}^3_{u_1,u_2,u_3}$ and for higher n by setting ${\mathbf{T}^d}_{3,\ldots,3} = \cdots = {\mathbf{T}^d}_{n,\ldots,n} = 1$. □

These extreme points can be used to generate other d-stochastic tensors as linear combinations. Due to the lemma above, we do not have the theoretical foundation to imply that hyperedges corresponding to the large entries in the scaled tensor must necessarily participate in a perfect matching. Nonetheless, the entries not in any perfect matching tend to become zero (not guaranteed for all though). For the worst case example of Karp-Sipser described above, the scaling indeed helps the entries corresponding to e_4, e_5 and e_6 to become zero.

Let $\mathbf{S}^3$ be the tensor obtained by swapping the 2nd and 3rd dimensions of $\mathbf{T}^3$. We can see that the tensor $\frac{1}{2}\mathbf{T}^3 + \frac{1}{2}\mathbf{S}^3$ has a perfect matching, however, obtained by a linear combination of two extreme points that are not permutation tensors. This shows that even if the heuristic selects an entry in the non-zero pattern of an extreme point without a perfect matching, we do not necessarily reduce our chances of obtaining a good matching because of the existence of entries outside the non-zero pattern of this extreme point.

On a d-partite, d-uniform hypergraph $H = (V, E)$, the Sinkhorn-Knopp algorithm used for scaling operates in iterations, each of which requires $\mathcal{O}(|E| \times d)$ time. In practice, we perform only a few iterations (e.g., 10–20). Since, we can match at most $|V|/d$ hyperedges, the overall run time cost associated with scaling is $\mathcal{O}(|V| \times |E|)$. A straightforward implementation of the second rule can take quadratic time in the worst case of a large number of repetitive merges with a given vertex. In practice, more of a linear time behavior should be observed for the second rule.

3.4 Hypergraph matching via pseudo scaling

In Algorithm 1, applying scaling at every step can be very costly. Here we propose an alternative idea inspired by the specifics of the Sinkhorn-Knopp algorithm to reduce the overall cost.

The Sinkhorn-Knopp algorithm scales a d-dimensional tensor $\mathbf{T}$ in a series of iterations by updating the set of vectors $x^{(1)}, \ldots, x^{(d)}$ where initially all values in all vectors are equal to 1. During an iteration, the coefficient vector $x^{(j)}$ for a given dimension j is updated by using

$$x_{i_j}^{(j)} = \frac{x_{i_j}^{(j)}}{\sum_{i_1,\ldots,i_{j-1},i_{j+1},\ldots,i_d} \left(\mathbf{T}_{i_1,\ldots,i_j,\ldots,i_d} \prod_{k=1}^{d} x_{i_k}^{(k)} \right)}, \text{for all } i_j \in \{1, \ldots, n_j\} \,. \tag{1}$$

These updates are done in a sequential order and for simplicity we assume that they happen in the dimension order: $1, \ldots, d$. Each vector entry $x_{i_j}^{(j)}$ corresponds to a vertex in the hypergraph. Let λ_{i_j} denote the degree of the vertex i_j from jth part. For the first iteration of (1), each $x_{i_1}^{(1)}$ is set to $\frac{1}{\lambda_{i_1}}$ since all values in the vectors are one. The pseudo scaling approach applies d parallel executions of updates (1) and sets each $x_{i_j}^{(j)} = \frac{1}{\lambda_{i_j}}$ for all $j \in \{1, \ldots, d\}$ and $i_j \in \{1, \ldots, n_j\}$. That is, each vertex gets a value inversely proportional to its degree. This avoids 10–20 iterations of Sinkhorn-Knopp and the $O(|E|)$ cost for each. However, as the name of the approach implies, this scaling is not exact.

With this approach each hyperedge $\{i_1, \ldots, i_d\}$ is associated with a value $\dfrac{1}{\prod_{j=1}^{d} \lambda_{i_j}}$. The selection procedure is the same as that of Algorithm 1, i.e., the edge with the maximum value is added to the matching set. We refer to this algorithm as Karp-Sipser-mindegree, as it selects a hyperedge based on a function of the degrees of the vertices. With a straightforward implementation, finding this hyperedge takes $O(|E|)$ time. For a better efficiency, the edges can be stored in a heap and when the degree of a node v decreases, the *increaseKey* heap operation can be called for all its edges.

3.5 Reduction to bipartite graph matching

A perfect matching in a d-partite, d-uniform hypergraph H remains perfect when projected on a $(d-1)$-partite, $(d-1)$-uniform hypergraph obtained by removing one of H's dimensions. Matchability in $(d-1)$-dimensional sub-hypergraphs has been investigated in [1] to provide an equivalent of Hall's Theorem for d-partite hypergraphs. These observations lead us to propose a heuristic called Bipartite-reduction. This heuristic tackles the d-partite, d-uniform case by recursively asking for matchings in $(d-1)$-partite, $(d-1)$-uniform hypergraphs and so on, until d=2.

Let us start with the case where $d = 3$. Let $G = (V_G, E_G)$ be the bipartite graph with the vertex set $V_G = V_1 \cup V_2$ obtained by deleting V_3 from a 3-partite, 3-regular hypergraph $H = (V, E)$. The edge $(u, v) \in E_G$ iff there exists a hyperedge $(u, v, z) \in E$. One can also assign a weight function $w(\cdot)$ to the edges during this step such as

$$w(u, v) = |\{z : (u, v, z) \in E\}| \; . \tag{2}$$

A maximum weighted (product, sum, etc.) matching algorithm can be used to obtain a matching M_G on G. A second bipartite graph $G' = (V_{G'}, E_{G'})$ is then created with $V_{G'} = (V_1 \times V_2) \cup V_3$ and $E_{G'} = \{(uv, z) : (u, v) \in M_G, (u, v, z) \in H\}$. Under this construction, any matching in G' corresponds to a valid matching in H. Furthermore, if the weight function (2) defined above is used the following holds.

Proposition 4. *Let $w(M_G) = \sum_{(u,v)\in M_G} w(u, v)$ be the size of the matching M_G found in G. Then G' has $w(M_G)$ edges.*

Proof. Consider a node $u \in V_1$ and let it be matched with $v \in V_2$ in M_G. The number of edges involving uv in G' is $|\{z : (u, v, z) \in E\}|$. We see that this number is equivalent to $w(u, v)$, and the result follows by considering each matched pair in M_G. $\square$

Thus, by selecting a maximum weighted matching M_G and maximizing $w(M_G)$, the largest number of edges will be kept in G'.

For d-dimensional matching, a similar process is followed. First, an ordering $i_1, i_2, \ldots, i_d$ of the dimensions is defined. At the jth bipartite reduction step, the matching is found between the dimension cluster $i_1 i_2 \cdots i_j$ and dimension i_{j+1} by similarly solving a bipartite matching instance where the edge $(u_1 \cdots u_j, v)$ exists iff vertices $u_1, \ldots, u_j$ were matched in previous steps and there exists an edge $(u_1, \ldots, u_j, v, z_{j+2}, \ldots, z_d)$ in H.

Unlike the previous heuristics, Bipartite-reduction does not have any approximation guarantee. We depict this with the following lemma.

Lemma 5. *If algorithms for the maximum cardinality or the maximum weighted matching (with the suggested edge weights (2)) problems are used, then Bipartite-reduction has a worst-case approximation ratio of $\Omega(n)$.*

Proof. We discuss initially the case for $d = 3$ and assume $n \geq 5$. Consider an $n \times n \times n$ hypergraph H with edges $e_i = (u_i, v_i, z_i), e'_i = (u_i, v_{1+i \bmod n}, z_2)$ and $e''_i = (u_i, v_{1+i \bmod n}, z_3)$ for $i \in \{1, \ldots, n\}$. There is a perfect matching containing all edges $e_1, \ldots, e_n$.

Suppose we create G by projecting the 3rd dimension. Then, the edges in G are either of the form $h_i = (u_i, v_i)$ with $w(h_i) = 1$ or $h'_i = (u_i, v_{1+i \bmod n})$ with $w(h'_i) = 2$. Both $\{h_1, \ldots, h_n\}$ and $\{h'_1, \ldots, h'_n\}$ form perfect matchings in G. If the weight function (2) is used, the algorithm will necessarily find the perfect matching $\{h'_1, \ldots, h'_n\}$. Otherwise, any matching algorithm can arbitrarily return $\{h'_1, \ldots, h'_n\}$.

Assuming that $\{h'_1, \ldots, h'_n\}$ is returned, the graph G' will have $2n$ edges. The edges will be either in the form $he_i = (u_i v_{1+i \mod n}, z_2)$ or $he'_i = (u_i v_{1+i \mod n}, z_3)$ for $i \in \{1, \ldots, n\}$. As seen, z_2 and z_3 are the only two vertices of the 3rd dimension which can be matched.

The algorithm will return a perfect matching, if we project a dimension other than the 3rd one. To extend H such that the approximation ratio is $\Omega(n)$ whichever dimension is projected, we need to introduce the following four additional set of edges: $e_i^{(3)} = (u_2, v_i, z_{1+i \mod n}), e_i^{(4)} = (u_3, v_i, z_{1+i \mod n}), e_i^{(5)} = (u_{1+i \mod n}, v_2, z_i)$ and $e_i^{(6)} = (u_{1+i \mod n}, v_3, z_i)$ for $i \in \{1, \ldots, n\}$ that mirror $\{e'_1, \ldots e'_n\}$ and $\{e''_1, \ldots, e''_n\}$. In this case, the maximum matching in G' will always be 5, as again the edges in $\{e_1, \ldots, e_n\}$ will be ignored.

The result holds for higher d by noting that H alongside its extension are valid 3-partite hypergraphs that can occur after a matching for vertices in dimensions $i_1, \ldots, i_{d-2}$ has been found. $\square$

3.6 Performing local search

A local search heuristic is proposed by Hurkens and Schrijver [24]. It starts from a feasible maximal matching M and performs a series of swaps until it is no longer possible. In a swap, k edges of M are replaced with at least $k+1$ new edges from $E \setminus M$ so that the cardinality of M increases by at least one. These k edges from M can be replaced with at most $d \times k$ new edges. Hence, these edges can be found by a polynomial algorithm enumerating all the possibilities. The approximation guarantee improves with higher k values. Local search algorithms are limited in practice due to their high time complexity. The algorithm might have to examine all $\binom{|M|}{k}$ subsets of M to find a feasible swap at each step. The algorithm by Cygan [9] which achieves a $\left(\frac{d+1+\varepsilon}{3}\right)$-approximation is based on a different swap scheme but is also not suited for large hypergraphs.

4 Experiments

To understand the relative performance of the proposed heuristics, we conducted a wide variety of experiments with both synthetic and real-life data. The experiments were performed on a computer equipped with intel Core i7-7600 CPU and 16GB RAM. We compare the adapted Greedy and Karp-Sipser heuristics with the proposed Karp-Sipser-scaling and Karp-Sipser-mindegree heuristics. For $d = 3$, we also consider a local search heuristic [24] called Local-Search, which replaces one hyperedge from a maximal matching M with at least two hyperedges from $E \setminus M$ to increase the cardinality of M. We did not consider local search schemes for higher dimensions or with better approximation ratios as they are computationally too expensive. For each hypergraph, we perform ten runs of Greedy and Karp-Sipser with different random decisions and take the maximum cardinality obtained. Since Karp-Sipser-scaling or Karp-Sipser-mindegree do not pick hyperedges randomly, we run them only once. We perform 20 steps of the scaling procedure in Karp-Sipser-scaling. We refer to quality of a matching M in a hypergraph H as the ratio of M's cardinality to the size of the smallest vertex partition of H.

		k					k		
	d	d^{d-3}	d^{d-2}	d^{d-1}		d	d^{d-3}	d^{d-2}	d^{d-1}
	2	-	0.87	1.00		2	-	0.84	1.00
	3	0.80	1.00	1.00		3	0.88	1.00	1.00
$n = 10$	4	1.00	1.00	1.00	$n = 30$	4	0.99	1.00	1.00
	5	1.00	1.00	1.00		5	$*$	1.00	1.00
	2	-	0.88	1.00		2	-	0.87	1.00
	3	0.85	1.00	1.00		3	0.84	1.00	1.00
$n = 20$	4	1.00	1.00	1.00	$n = 50$	4	$*$	1.00	1.00
	5	1.00	1.00	1.00		5	$*$	$*$	$*$

Table 1 – The average maximum matching cardinalities of five random instances over n on random k-out, d-partite, d-uniform hypergraphs for different k, d, and n. No runs for $k = d^{d-3}$ for $d = 2$, and the problems marked with $*$ were not solved within 24 hours.

4.1 Experiments on random hypergraphs

We perform experiments on two classes of d-partite, d-uniform random hypergraphs where each part has n vertices. The first class contains random k-out hypergraphs, and the second one contains sparse random hypergraphs.

Random k-out, d-partite, d-uniform hypergraphs
Here, we consider random k-out, d-partite, d-uniform hypergraphs described in Section 2. Hence (ignoring the duplicate ones), these hypergraphs have around $d \times k \times n$ hyperedges. These k-out, d-partite, d-uniform hypergraphs have been recently analyzed in the matching context by Devlin and Kahn [11]. They state in passing that k should be exponential in d for a perfect matching to exist with high probability. The bipartite graph variant of the same problem, i.e., with $d = 2$, has been extensively studied in the literature [17, 25, 34]; a perfect matching almost always exists in a random 2-out bipartite graph [34].

We first investigate the existence of perfect matchings in random k-out, d-partite, d-uniform hypergraphs. For this purpose, we implemented the linear program of d-dimensional matching in CPLEX and found the maximum cardinality of a matching in k-out hypergraphs with $k \in \{d^{d-3}, d^{d-2}, d^{d-1}\}$ for $d \in \{2, \dots, 5\}$ and $n \in \{10, 20, 30, 50\}$. For each (k, d, n) triple, we created five hypergraphs and computed their maximum cardinality matchings. For $k = d^{d-3}$, we encountered several hypergraphs with no perfect matching, especially for $d = 3$. The hypergraphs with $k = d^{d-2}$ were also lacking a perfect matching for $d = 2$. However, all the hypergraphs we created with $k = d^{d-1}$ had at least one. Based on these results, we experimentally confirm Devlin and Kahn's statement. We also conjecture that d^{d-1}-out random hypergraphs have perfect matchings almost surely. The average maximum matching cardinalities we obtained in this experiment are given in Table 1. In this table, we do not have results for $k = d^{d-3}$ for $d = 2$, and the cases marked with $*$ were not solved within 24 hours.

We now compare the performance of the proposed heuristics on random k-out, d-partite, d-uniform hypergraphs $d \in \{3, 6, 9\}$ and $n \in \{1000, 10000\}$. We tested with k values equal to powers of 2 for $k \leq d \log d$. The results are summarized in Figure 1. For each (k, d, n) triplet, we create ten random instances and present the average performance of the heuristics on them. The x-axis in each figure denotes k, and the y-axis reports the matching cardinality over n. As seen, Karp-Sipser-scaling and Karp-Sipser-mindegree have the best performance, comfortably beating the other alternatives. For $d = 3$ Karp-Sipser-scaling dominates Karp-Sipser-mindegree, but when $d > 3$ we see that Karp-Sipser-mindegree has the best performance. Karp-Sipser performs better than Greedy. However, their performances get closer as d increases. This is due to the fact that the conditions for Rule-1 and Rule-2 hold less often for larger d as we have more restrictions to encounter in such cases. Bipartite-reduction has worse performance than the others, and the gap in the performance grows as d increases. This happens, since at each step, we impose more and more conditions on the edges involved and there is no chance to recover from bad decisions.

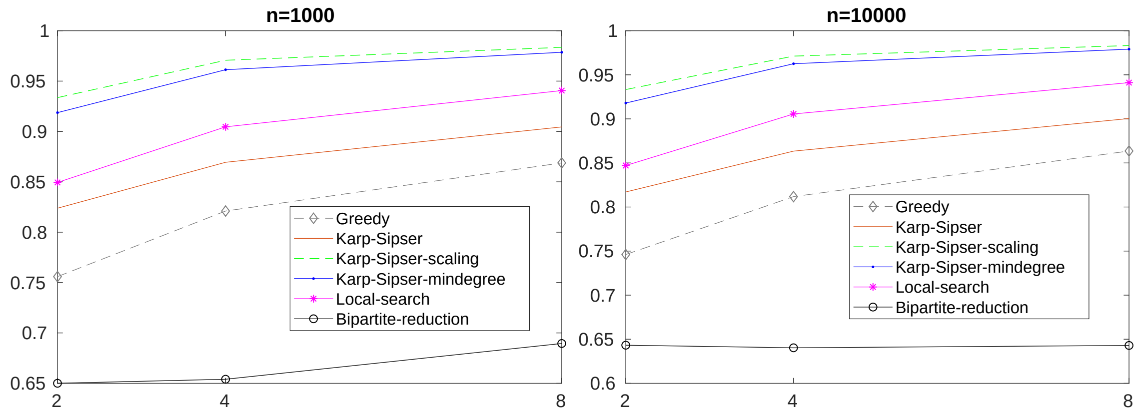
Sparse random d-partite, d-uniform hypergraphs
Here, we consider a random d-partite, d-uniform hypergraph H_i is created with $i \times n$ hyperedges. The parameters used for this experiment are $i \in \{1, 3, 5, 7\}$, $n \in \{4000, 8000\}$, and $d \in \{3, 6, 9\}$. Each H_i is created by choosing the vertices of a hyperedge uniformly at random for each dimension. We do not allow duplicate hyperedges. Another random hypergraph H_{i+M} is then obtained by planting a perfect matching to H_i. We again generate ten random instances for each parameter setting. We do not present results for Bipartite-reduction as it was always worse than the others, as before. The average quality of different heuristics on these instances is shown in Figure 2. The experiments confirm that Karp-Sipser performs consistently better than Greedy. Furthermore, Karp-Sipser-scaling performs significantly better than Karp-Sipser. Karp-Sipser-scaling works even better than the local search heuristic, and it is the only heuristic that is capable of finding planted perfect matchings for a significant number of the runs. In particular when $d > 3$, it finds a perfect matching on H_{i+M}'s in all cases except for when $d = 6$ and $i = 7$. For $d = 3$, it finds a perfect matching only when $i = 1$ and attains a near perfect matching when $i = 3$. Interestingly Karp-Sipser-mindegree outperforms Karp-Sipser-scaling on H_is but is dominated on H_{i+M}s, where it is the second best performing heuristic.

4.2 Evaluating algorithmic choices

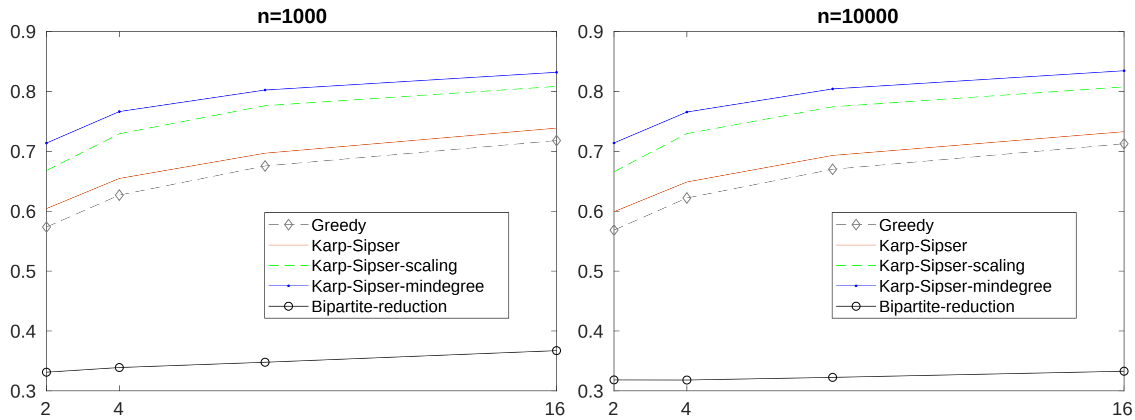
Here, we evaluate the use of scaling and the importance of Rule-1 and Rule-2.

Scaling vs no-scaling
To evaluate and emphasize the contribution of scaling better, we compare the performance of the heuristics on a particular family of d-partite, d-uniform hypergraphs where their bipartite counterparts have been used before as challenging instances for the original Karp-Sipser^g heuristic [12].

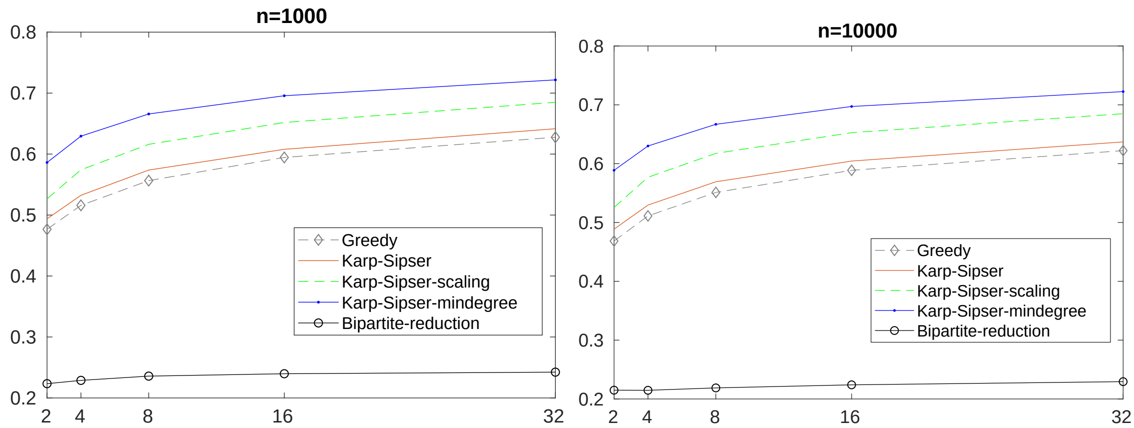
Let $\mathbf{A}_{KS}$ be an $n \times n$ matrix. Let R_1 and C_1 be $\mathbf{A}_{KS}$'s first $n/2$ rows and columns, respectively, and R_2 and C_2 be the remaining $n/2$ rows and columns, respectively. Let the block $R_1 \times C_1$ be full and the block $R_2 \times C_2$ be empty. A perfect bipartite graph matching is hidden inside the blocks $R_1 \times C_2$ and $R_2 \times C_1$ by introducing a non-zero diagonal to each. In addition, a parameter t connects the last t rows of R_1 with all the columns in C_2. Similarly, the last t columns in C_1 are connected to all the rows in R_2. An instance from this family of matrices is depicted in Figure 3. Karp-Sipser^g is impacted negatively when $t \geq 2$ whereas Greedy^g struggles even with $t = 0$ because random edge selections will almost always be from the dense $R_1 \times C_1$ block. To adapt this scheme to hypergraphs/tensors, we generate a 3-dimensional tensor $\mathbf{T}_{KS}$ such that the nonzero pattern of each marginal of the 3rd dimension is identical to that of $\mathbf{A}_{KS}$. Table 2 shows the performance of the heuristics (i.e., matching cardinality normalized with n) for 3-dimensional tensors with $n = 300$ and $t \in \{2, 4, 8, 16, 32\}$.



(a) $d = 3$, $n = 1000$ (left) and $n = 10000$ (right)



(b) $d = 6$, $n = 1000$ (left) and $n = 10000$ (right)



(c) $d = 9$, $n = 1000$ (left) and $n = 10000$ (right)

Figure 1 – The performance of the heuristics on k-out, d-partite, d-uniform hypergraphs with n vertices at each part. The y-axis is the ratio of matching cardinality to n whereas the x-axis is k. No Local-Search for $d = 6$ and $d = 9$.

	H_i: Random Hypergraph										H_{i+M}: Random Hypergraph with Perfect Matching									
	Greedy		Local Search		Karp-Sipser		Karp-Sipser-scaling		Karp-Sipser-minDegree		Greedy		Local Search		Karp-Sipser		Karp-Sipser-scaling		Karp-Sipser-minDegree	
i	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000
1	0.43	0.42	0.47	0.47	0.49	0.48	0.49	0.48	0.49	0.48	0.75	0.75	0.93	0.93	1.00	1.00	1.00	1.00	1.00	1.00
3	0.63	0.63	0.71	0.71	0.73	0.72	0.76	0.76	0.78	0.77	0.72	0.71	0.82	0.81	0.81	0.81	0.99	0.99	0.92	0.92
5	0.70	0.70	0.80	0.80	0.78	0.78	0.86	0.86	0.88	0.88	0.75	0.74	0.84	0.84	0.82	0.82	0.94	0.94	0.92	0.92
7	0.75	0.75	0.84	0.84	0.81	0.81	0.94	0.94	0.93	0.93	0.77	0.77	0.87	0.87	0.83	0.83	0.96	0.96	0.94	0.94

(a) $d = 3$, without (left) and with (right) the planted matching

	H_i: Random Hypergraph								H_{i+M}: Random Hypergraph with Perfect Matching							
	Greedy		Karp-Sipser		Karp-Sipser-scaling		Karp-Sipser-minDegree		Greedy		Karp-Sipser		Karp-Sipser-scaling		Karp-Sipser-minDegree	
i	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000
1	0.31	0.31	0.35	0.35	0.35	0.35	0.36	0.37	0.62	0.61	0.90	0.89	1.00	1.00	1.00	1.00
3	0.43	0.43	0.47	0.47	0.48	0.48	0.54	0.54	0.51	0.50	0.56	0.55	1.00	1.00	0.99	0.99
5	0.48	0.48	0.52	0.52	0.54	0.54	0.61	0.61	0.52	0.52	0.56	0.55	1.00	1.00	0.97	0.97
7	0.52	0.52	0.55	0.55	0.59	0.59	0.66	0.66	0.54	0.54	0.57	0.57	0.84	0.80	0.71	0.70

(b) $d = 6$, without (left) and with (right) the planted matching

	H_i: Random Hypergraph								H_{i+M}: Random Hypergraph with Perfect Matching							
	Greedy		Karp-Sipser		Karp-Sipser-scaling		Karp-Sipser-minDegree		Greedy		Karp-Sipser		Karp-Sipser-scaling		Karp-Sipser-minDegree	
i	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000	4000	8000
1	0.25	0.24	0.27	0.27	0.27	0.27	0.30	0.30	0.56	0.55	0.80	0.79	1.00	1.00	1.00	1.00
3	0.34	0.33	0.36	0.36	0.36	0.36	0.43	0.43	0.40	0.40	0.44	0.44	1.00	1.00	0.99	1.00
5	0.38	0.37	0.40	0.40	0.41	0.41	0.48	0.48	0.41	0.40	0.43	0.43	1.00	1.00	0.99	0.99
7	0.40	0.40	0.42	0.42	0.44	0.44	0.51	0.51	0.42	0.42	0.44	0.44	1.00	1.00	0.97	0.96

(c) $d = 9$, without (left) and with (right) the planted matching

Figure 2 – Performance comparisons on d-partite, d-uniform hypergraphs with $n = \{4000, 8000\}$. H_i contains $i \times n$ random hyperedges, and H_{i+M} contains an additional perfect matching.

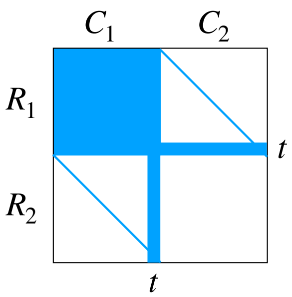


Figure 3 – $\mathbf{A}_{KS}$: A challenging instance for Karp-Sipser[g].

t	Greedy	Local Search	Karp-Sipser	Karp-Sipser-scaling	Karp-Sipser-minDegree
2	0.53	0.99	0.53	1.00	1.00
4	0.53	0.99	0.53	1.00	1.00
8	0.54	0.99	0.55	1.00	1.00
16	0.55	0.99	0.56	1.00	1.00
32	0.59	0.99	0.59	1.00	1.00

Table 2 – Performance of the proposed heuristics on 3-partite, 3-uniform hypergraphs corresponding to $\mathbf{T}_{KS}$ with $n = 300$ vertices in each part.

	d					
	2		3		6	
n	quality	$\frac{r}{n}$	quality	$\frac{r}{n}$	quality	$\frac{r}{n}$
1000	0.83	0.45	0.85	0.47	0.80	0.31
2000	0.86	0.53	0.87	0.56	0.80	0.30
4000	0.82	0.42	0.75	0.17	0.84	0.45

Table 3 – Quality of matching and the number r of the applications of Rule-1 over n in Karp-Sipser$_{R_1}$, for hypergraphs corresponding to $\mathbf{T}_{RF}$. Karp-Sipser obtains perfect matchings.

The use of scaling indeed reduces the influence of the misleading hyperedges in the dense block $R_1 \times C_1$, and the proposed Karp-Sipser-scaling heuristic always finds the perfect matching as does Karp-Sipser-mindegree. However, Greedy and Karp-Sipser perform significantly worse. Furthermore, Local Search returns a 0.99-approximation in every case because it ends up in a local optima.

Rule-1 vs Rule-2

We finish the discussion on the synthetic data by focusing on Karp-Sipser. Recall from Section 3.2 that Karp-Sipser has two rules. In the bipartite case, a variant of Karp-Sipser^g in which Rule-2 is not applied received more attention than the original version, because it is simpler to implement and easier to analyze. This simpler variant has been shown to obtain good results both theoretically [27] and experimentally [12]. Recent work [2] shows that both rules are needed to obtain perfect matchings in random cubic graphs.

We present a family of hypergraphs to demonstrate that using Rule-2 leads to better performance than using Rule-1 only. We use Karp-Sipser$_{R_1}$ to refer to Karp-Sipser without Rule-2. As before, we describe first the bipartite case. Let $\mathbf{A}_{RF}$ be a $n \times n$ matrix with $(\mathbf{A}_{RF})_{i,j} = 1$ for $1 \leq i \leq j \leq n$, and $(\mathbf{A}_{RF})_{2,1} = (\mathbf{A}_{RF})_{n,n-1} = 1$. That is $\mathbf{A}_{RF}$ is composed of an upper triangular matrix and two additional subdiagonal nonzeros. The first two columns and the last two rows have two nonzeros. Assume without loss of generality that the first two rows are merged by applying Rule-2 on the first column (which is discarded). Then in the reduced matrix, the first column (corresponding to the second column in the original matrix) will have one nonzero. Rule-1 can now be applied whereupon the first column in the reduced matrix will have degree one. The process continues similarly until the reduced matrix is a 2×2 dense block, where applying Rule-2 followed by Rule-1 yields a perfect matching. If only Rule-1 reductions are allowed, initially no reduction can be applied and randomly chosen edges will be matched, which negatively affects the quality of the returned matching.

For higher dimensions we proceed as follows. Let $\mathbf{T}_{RF}$ be a d-dimensional $n \times \cdots \times n$ tensor. We set $(\mathbf{T}_{RF})_{i,j,\ldots,j} = 1$ for $1 \leq i \leq j \leq n$ and $(\mathbf{T}_{RF})_{1,2,\ldots,2} = (\mathbf{T}_{RF})_{n,n-1,\ldots,n-1} = 1$. By similar reasoning, we see that Karp-Sipser with both reduction rules will obtain a perfect matching, whereas Karp-Sipser$_{R_1}$ will struggle. We give some results in Table 3 that show the difference between the two. We test for $n \in \{1000, 2000, 4000\}$ and $d \in \{2, 3, 6\}$, and show the quality of Karp-Sipser$_{R_1}$ and the number of times that Rule-1 is applied over n. We present the best result over 10 runs.

As seen in Table 3, Karp-Sipser$_{R_1}$ obtains matchings that are about 13–25% worse than Karp-Sipser. Furthermore, the larger the number of Rule-1 applications is, the higher the quality is.

Tensor	d	Dimensions	nnz	Greedy	Local-Search	Karp-Sipser	Karp-Sipser-minDegree	Karp-Sipser-scaling	Bipartite-Reduction
Uber	3	$183 \times 1140 \times 1717$	1,117,629	183	183	183	183	183	183
nips [19]	3	$2,482 \times 2,862 \times 14,036$	3,101,609	1,847	1,991	1,839	2005	2,007	2,007
Nell-2 [5]	3	$12,092 \times 9,184 \times 28,818$	76,879,419	3,913	4,987	3,935	5,100	5,154	5,175
Enron [31]	4	$6,066 \times 5,699 \times 244,268 \times 1,176$	54,202,099	875	-	875	988	1,001	898

Table 4 – Four real-life tensors and the performance of the proposed heuristics on the corresponding hypergraphs.No result for Local-Search for Enron, as it is four dimensional.

4.3 Experiments with real-life tensor data

We also evaluate the performance of the proposed heuristics on some real-life tensors selected from the FROSTT library [33]. The descriptions of the tensors are given in Table 4. For nips and uber, a dimension of size 17 and 24 is dropped respectively since they restrict the size of maximum cardinality matching As described before, a d-partite, d-uniform hypergraph is obtained from a d-dimensional tensor by keeping a vertex for each dimension index, and a hyperedge for each nonzero. Unlike the previous hypergraphs in this section, the parts of the hypergraphs obtained from Unlike the previous experiments, the parts of the hypergraphs obtained from real-life tensors in Table 4 do not have an equal number of vertices. In this case, although the scaling algorithm works along the same lines, it's output is slightly different. Let $n_i = |V_i|$ be the cardinality at ith dimension and $n_{max} = \max_{1 \leq i \leq d} n_i$ be the maximum one. By slightly modifying Sinkhorn-Knopp, for each iteration of Karp-Sipser-scaling, we scale the tensor such that the marginals in dimension i sum up to n_{max}/n_i instead of one. The results in Table 4 resemble those from previous sections; Karp-Sipser-scaling has the best performance and is slightly superior to Karp-Sipser-mindegree. Greedy and Karp-Sipser are close to each other and when it is feasible, Local-Search is better than them. In addition we see that in these instances Bipartite-reduction exhibits a good performance: its performance is at least as good as Karp-Sipser-scaling for the first three instances, but about 10% worse for the last one.

4.4 Experiments with an independent set solver

We compare Karp-Sipser-scaling and Karp-Sipser-mindegree with the idea of reducing MAX-d-DM to the problem of finding an independent set in the line graph of the given hypergraph. We show that this transformation can lead to good results, but is restricted because line graphs can require too much space.

We use KaMIS [29] to find independent sets in graphs. KaMIS uses a plethora of reductions and a genetic algorithm in order to return high cardinality independent sets. We use the default settings of KaMIS (where execution time is limited to 600 seconds) and generate the line graphs with efficient sparse matrix–matrix multiplication routines. We run KaMIS, Greedy, Karp-Sipser-scaling, and Karp-Sipser-mindegree on a few hypergraphs from previous tests. The results are summarized in Table 5. The run time of Greedy was less than one second in all instances. KaMIS operates in rounds, and we give the quality and the run time of the first round and the final output. We note that KaMIS considers the time-limit only after the first round has been completed. As can be seen, while the quality of KaMIS is always good and in most cases superior to Karp-Sipser-scaling and

	KaMIS						Greedy	Karp-Sipser-scaling		Karp-Sipser-mindegree	
	line graph	Round 1		Output							
hypergraph	gen. time	quality	time	quality	time		quality	quality	time	quality	time
8-out, $n = 1000$, $d = 3$	10	0.98	80	0.99	600		0.86	0.98	1	0.98	1
8-out, $n = 10000$, $d = 3$	112	0.98	507	0.99	600		0.86	0.98	197	0.98	1
8-out, $n = 1000$, $d = 9$	298	0.67	798	0.69	802		0.55	0.62	2	0.67	1
$n = 8000$, $d = 3$, H_3	1	0.77	16	0.81	602		0.63	0.76	5	0.77	1
$n = 8000$, $d = 3$, H_{3+M}	2	0.89	25	1.00	430		0.70	1.00	11	0.91	1

Table 5 – Run time (in seconds) and performance comparisons between KaMIS, Greedy, and Karp-Sipser-scaling. The time required to create the line graphs should be added to KaMIS's overall time.

Karp-Sipser-mindegree, it is also significantly slower (its principle is to deliver high quality results). We also observe that the pseudo scaling of Karp-Sipser-mindegree indeed helps to reduce the run time compared to Karp-Sipser-scaling.

The line graphs of the real-life instances from Table 4 are too large to be handled. We estimate using known techniques [7] the number of edges in these graphs to range from 1.5×10^{10} to 4.7×10^{13}. The memory needed ranges from 126GB to 380TB if edges are stored twice (assuming 4 bytes per edge).

5 Conclusion and future work

We have proposed heuristics for the Max-d-DM problem by generalizing existing heuristics for the maximum cardinality matching in bipartite graphs. The experimental analysis on various hypergraphs/tensors show the effectiveness and efficiency of the proposed heuristics. As future work, we plan to investigate the stated conjecture that d^{d-1}-out random hypergraphs have perfect matchings almost always, and analyze the theoretical guarantees of the proposed algorithms.

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